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Nonlinear Evolution of Two-Magnetofluid Instability

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The nonlinear surface instability of a horizontal interface separating two magnetic fluids of different densities, magnetic permeabilities, and velocities, including surface tension effects, is investigated. The magnetic field is applied along the direction of streaming. It is shown that the evolution of the amplitude is governed by a nonlinear Ginzburg–Landau equation with the use of the multiple scale method. When the influence of streaming is neglected, the nonlinear diffusion equation is obtained. Further, it is shown that a nonlinear Schrödinger equation is obtained in the absence of gravity. The various stability criteria are discussed from these equations, of both Rayleigh–Taylor and Kelvin–Helmholtz problems, both analytically and numerically and the stability diagrams are obtained. Obtained also are the stability properties of solitary solutions to the Ginzburg–Landau equation in the case of constant surface tension.

1. INTRODUCTION

As described by Rosensweig (1985) and Zahn and Rosensweig (1991), magnetic fluids are synthesized by colloidally suspending solid magnetic particles of subdomain size. The particles do not separate out from the liquid carrier, as they are kept in constant agitation by random thermal molecular motion. Magnetic fluids are described by a magnetic permeability larger than the magnetic permeability of free space. There is a magnetization force on the magnetic fluid in the presence of a nonuniform magnetic field whereby high-permeability material is attracted to high-magnetic-field regions.

A magnetic fluid behaves like a fluid possessing magnetic properties in an applied magnetic field. By subjecting magnetic fluids to nonuniform

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magnetic fields, it is possible to control the liquid pressure, the shape of the fluid interface, and the flow pattern.

In order to apply these features of magnetic fluids to fluid engineering, many proposals have been made to develop new devices which can utilize such magnetic fluids. Zelazo and Melcher (1969) examined theoretically as well as experimentally the plane wave propagation for two superposed magnetic fluids in the presence of a tangential field, and demonstrated that the magnetic field exerts a stabilizing influence on waves. In their investigation of the non-linear Kelvin–Helmholtz instability in magnetic fluids, Malik and Singh (1986) showed that the wave train solution of constant amplitude is unstable against modulation if the product of the group velocity rate and the nonlinear interaction coefficient is negative. However, in the studies mentioned above, the surface tension taken was constant.

The effect of the variable surface tension on the motion of the fluids has been extensively treated by many authors. The linear analysis of the surface tension effect was presented by Pearson (1958) and the conditions for the onset of the instability were derived. Levich (1962) gave a more complete discussion, showing how the variation of surface tension is fixed by the variation of surfactant concentration around the interface which is fixed by the balance between adsorption and desorption of the surfactant from the liquid, and convection and diffusion along the interface. Characterization of surfactant adsorption at fluid–fluid interfaces with respect to both the equilibrium and dynamic behavior is essential to a complete understanding of mass transfer. Many important phenomena, such as interfacial turbulence, thin-film stability, and retardation of drop motion, are consequences of the fact that the surface tension varies with the interfacial concentration [see, for example, Scriven and Sternling (1960) and Sørensen (1978)].

Weakly nonlinear aspects of the Marangoni effect, gradient in surface tension, based on amplitude expansions were studied by Sivashinsky (1982), Funada (1987), Elhefnawy (1990), and Oron and Rosenau (1992). Those studies differ from each other in asymptotic representations of the independent and the dependent variables and lead to different evolution equations describing the behavior of the interface in different parametric regimes.

In this presentation, we study the nonlinear surface instability at the interface of two semi-infinite superposed magnetic fluids, taking into account the effect of the surface tension. The fluids are moving with uniform speeds parallel to the common interface and subjected to the tangential magnetic field. In Section 2, we formulate the problem and derive the characteristic equation for the first order and the solvability conditions for the higher orders, using the multiple-scales method.

The linear stability theory is studied for both Rayleigh–Taylor and Kelvin–Helmholtz problems. From the second- and third-order theories developed in Section 3, we have derived a nonlinear Ginzburg–Landau equation. From this equation, the various stability criteria are obtained for both Rayleigh–Taylor and Kelvin–Helmholtz problems. Finally, the stability properties of solitary solutions in the case of constant surface tension are discussed in Section 4.

2. THE BASIC EQUATIONS AND THE MULTIPLE-SCALES METHOD

Consider two inviscid, incompressible, superposed magnetic fluids separated by an interface $z = 0$. The half-space $z < 0$ is occupied by the magnetic fluid of density ρ_1 and magnetic permeability μ_1, whereas the region $z > 0$ contains the magnetic fluid of density ρ_2 and magnetic permeability μ_2. The lower and the upper fluids are streaming with velocities U_1 and U_2 along the positive x direction, respectively. The magnetic field H_0 acts along the direction of the flow. Both fluids are assumed to be homogeneous and the motion in them is irrotational. The analysis takes into account the surface tension σ as well as a gravitational force per unit mass g acting normal to the interface and directed in the negative z direction. We assume that the surface tension is a function of the adsorption Γ, i.e., $\sigma = \sigma_0 - \alpha(\Gamma - \Gamma_0)$, where $\Gamma_0 > 0$ is the unperturbed adsorption, σ_0 is the surface tension at Γ_0, and $\alpha = -d\sigma/d\Gamma > 0$ is the negative rate of change of surface tension with adsorption.

The basic equations which govern the system are

$$\nabla^2\phi_j = 0 \qquad (j = 1, 2) \tag{1}$$

$$\nabla^2\psi_j = 0 \qquad (j = 1, 2) \tag{2}$$

where $j = 1$ denotes the region $z < 0$, and $j = 2$ the region $z > 0$. Here ϕ and ψ are the velocity potential and the magnetic potential, respectively.

Since the motion must vanish from the interface, we must have

$$\nabla(\phi_j, \psi_j) \to 0 \qquad \text{as} \quad |z| \to \infty$$

The kinematic condition at the interface is

$$\frac{\partial\eta}{\partial t} + U_j\frac{\partial\eta}{\partial x} - \frac{\partial\phi_j}{\partial z} + \frac{\partial\eta}{\partial x}\frac{\partial\phi_j}{\partial x} + \frac{\partial\eta}{\partial y}\frac{\partial\phi_j}{\partial y} = 0 \qquad \text{at} \quad z = \eta(x, y, t) \tag{3}$$

where $\eta(x, y, t)$ stands for the elevation of the interface.

The mass balance condition at the interface is

$$\frac{d\Gamma}{dt}+\frac{\Gamma}{2a}\frac{da}{dt}-\Gamma\frac{\partial^2\phi_1}{\partial z^2}-D\left(\frac{\partial^2\Gamma}{\partial x^2}+\frac{\partial^2\Gamma}{\partial y^2}\right)=0 \qquad \text{at} \quad z=\eta(x,y,t) \qquad (4)$$

where $a=1+(\partial\eta/\partial x)^2+(\partial\eta/\partial y)^2$ is the surface metric determinant and D is the diffusion coefficient.

The continuity of the normal and tangential components of the magnetic field across the interface requires

$$\mu H_{1n}=H_{2n} \qquad \text{at} \quad z=\eta(x,y,t) \qquad (5)$$

$$H_{1t}=H_{2t} \qquad \text{at} \quad z=\eta(x,y,t) \qquad (6)$$

Since the normal stress across the interface must be continuous, we obtain

$$\begin{aligned} & g\tilde{\Gamma}+g\eta(\rho_1-\rho_2)+\rho_1\left(\frac{\partial\phi_1}{\partial t}+U_1\frac{\partial\phi_1}{\partial x}\right)-\rho_2\left(\frac{\partial\phi_2}{\partial t}+U_2\frac{\partial\phi_2}{\partial x}\right) \\ & +\frac{1}{2}[\rho_1(\nabla\phi_1)^2-\rho_2(\nabla\phi_2)^2]+\Gamma\frac{d}{dt}\left(\frac{\partial\phi_1}{\partial z}\right) \\ & +\frac{1}{2}\Gamma(\nabla\eta)^2\left(g+\frac{\partial^2\phi_1}{\partial t\partial z}+U_1\frac{\partial^2\phi_1}{\partial x\partial z}\right) \\ & -(\sigma_0-\alpha\tilde{\Gamma})\left[1+\left(\frac{\partial\eta}{\partial x}\right)^2+\left(\frac{\partial\eta}{\partial y}\right)^2\right]^{-3/2}\left\{\frac{\partial^2\eta}{\partial x^2}\left[1+\left(\frac{\partial\eta}{\partial y}\right)^2\right]\right. \\ & \left.-2\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}\frac{\partial^2\eta}{\partial x\partial y}+\frac{\partial^2\eta}{\partial y^2}\left[1+\left(\frac{\partial\eta}{\partial x}\right)^2\right]\right\} \\ & =\frac{\mu_1-\mu_2}{8\pi}(H_{1t}^2+\mu H_{1n}^2) \qquad \text{at} \quad z=\eta(x,y,t) \qquad (7) \end{aligned}$$

where $\mu=\mu_1/\mu_2$ and $\tilde{\Gamma}=\Gamma-\Gamma_0$. Here H_n and H_t represent the normal and the tangential components of the magnetic field, respectively.

As the boundary conditions (3)–(7) are given at the free surface $z=\eta(x,y,t)$, one needs prior information about $\eta(x,y,t)$. To surmount this difficulty, we use Maclaurin's expansions about $z=0$ of the physical quantities appearing in equations (3)–(7), thereby reducing the conditions at the unperturbed levels $z=0$.

To obtain the asymptotic solution to the system of equations (1)–(7), we introduce the three sets of slow variables

$$(x_n,y_n,t_n)=\varepsilon^n(x,y,t), \qquad n=0,1,2 \qquad (8)$$

where ε is a small dimensionless parameter representing the size of the perturbations. The method we use is that of multiple scales, which relies on the fact that the wave amplitude is being modulated slowly in space and time. We assume that

$$\Phi(x, y, t) = \sum_{n=1}^{3} \varepsilon^n \Phi_n(x_0, x_1, x_2, y_0, y_1, y_2, t_0, t_1, t_2) + O(\varepsilon^4) \tag{9}$$

$$\Psi(x, y, z, t) = \sum_{n=1}^{3} \varepsilon^n \Psi_n(x_0, x_1, x_2, y_0, y_1, y_2, z, t_0, t_1, t_2) + O(\varepsilon^4) \tag{10}$$

where Φ can be either of the physical quantities η or $\tilde{\Gamma}$; while Ψ can be either ϕ_j or ψ_j; we take into account that η_1 is expanded in the form

$$\eta_1 = A(x_1, x_2, y_1, y_2, t_1, t_2) \exp[i(K_x x_0 + K_y y_0 - wt_0)] + \bar{A}(x_1, x_2, y_1, y_2, t_1, t_2) \exp[-i(K_x x_0 + K_y y_0 - wt_0)] \tag{11}$$

The bar denotes the complex conjugate; A is a slowly varying amplitude to be determined later by the solvability conditions; $(K_x^2 + K_y^2)^{1/2} = K$ is a wavenumber which should be a real and positive number; and w is the frequency of the disturbance.

The expansions (8)–(10) are uniformly valid for $-\infty < x < \infty$, $-\infty < y < \infty$, and $0 < t < \infty$. On substitution in equations (1)–(7), we get the linear and the successive nonlinear partial differential equations of the various orders. The solution of the problem in any order can be deduced with knowledge of the solutions of all the previous orders. The procedure is straightforward but lengthy and it will not be included here. The details are available from the author and are outlined by Nayfeh (1976).

The first-order problem leads to the characteristic equation

$$F(w, K_x, K_y) = g(\rho_1 - \rho_2) + \sigma_0 K^2 - \left(\Gamma_0 + \frac{\rho_1}{K}\right)(w - K_x U_1)^2 - \frac{\rho_2}{K}(w - K_x U_2)^2 + \frac{g\Gamma_0 K(w - K_x U_1)}{w - K_x U_1 + iK^2 D} + \frac{H_0^2 K_x^2 (\mu_1 - \mu_2)^2}{4\pi K(\mu_1 + \mu_2)} = 0 \tag{12}$$

which is similar to the results of Rosensweig (1985) in the limit when Γ_0 is neglected.

If we carry on the problem to the second-order set of equations, we may substitute the solutions of the first-order problem into the second-order one and solve the resulting equations. The solutions yield the solvability condition

$$-\frac{\partial F}{\partial w}\frac{\partial A}{\partial t_1} + \frac{\partial F}{\partial K_x}\frac{\partial A}{\partial x_1} + \frac{\partial F}{\partial K_y}\frac{\partial A}{\partial y_1} = 0 \tag{13}$$

while the third-order problem implies the condition

$$i\left(-\frac{\partial F}{\partial w}\frac{\partial A}{\partial t_2}+\frac{\partial F}{\partial K_x}\frac{\partial A}{\partial x_2}+\frac{\partial F}{\partial K_y}\frac{\partial A}{\partial y_2}\right)$$
$$+\frac{1}{2}\frac{\partial^2 F}{\partial w^2}\frac{\partial^2 A}{\partial t_1^2}-\frac{\partial^2 F}{\partial w\partial K_x}\frac{\partial^2 A}{\partial x_1\partial t_1}$$
$$-\frac{\partial^2 F}{\partial w\partial K_y}\frac{\partial^2 A}{\partial y_1\partial t_1}+\frac{1}{2}\frac{\partial^2 F}{\partial K_x^2}\frac{\partial^2 A}{\partial x_1^2}+\frac{\partial^2 F}{\partial K_x\partial K_y}\frac{\partial^2 A}{\partial x_1\partial y_1}$$
$$+\frac{1}{2}\frac{\partial^2 F}{\partial K_y^2}\frac{\partial^2 A}{\partial y_1^2}=JA^2\bar{A} \tag{14}$$

where

$$J=2\Lambda\Big\{(\rho_1+3\Gamma_0 K)(w-K_xU_1)^2-\rho_2(w-K_xU_2)^2-g\Gamma_0K^2$$
$$+\frac{1}{2}g\Gamma_0K^2(w-K_xU_1)(\gamma_1+4\gamma_2+\gamma_3)$$
$$-\Gamma_0K(w-K_xU_1)(\gamma_2+2\gamma_3)[\alpha K^2+(w-K_xU_1)^2]$$
$$+\frac{K_x^2H_0^2(\mu_2-\mu_1)^3}{4\pi(\mu_2+\mu_1)^2}\Big\}+2K[\rho_1(w-K_xU_1)^2+\rho_2(w-K_xU_2)^2]$$
$$-\frac{3}{2}\sigma_0K^4+\frac{1}{2}\Gamma_0K^2(w-K_xU_1)^2-\frac{KK_x^2H_0^2(\mu_1-\mu_2)^2}{2\pi(\mu_1+\mu_2)}$$
$$+\Gamma_0K^2(w-K_xU_1)^3(2\gamma_1+3\gamma_3)$$
$$+\Gamma_0K^2(w-K_xU_1)(\gamma_1-\gamma_2)[\alpha K^2+(w-K_xU_1)^2]$$
$$+g\Gamma_0K^3\gamma_1^2(w-K_xU_1)^2-g\Gamma_0K^3(w-K_xU_1)(8.5\gamma_1-2\gamma_2+\gamma_3) \tag{15}$$

with

$$\Lambda=\Big\{\rho_2(w-K_xU_2)^2-(\rho_1+3\Gamma_0K)(w-K_xU_1)^2-g\Gamma_0\gamma_2K^2(w-K_xU_1)$$
$$+\Gamma_0\gamma_1K(w-K_xU_1)[gK+\alpha K^2+(w-K_xU_1)^2]$$
$$+\frac{1}{2}g\Gamma_0K^2-\frac{K_x^2H_0^2(\mu_2-\mu_1)^3}{4\pi(\mu_2+\mu_1)^2}\Big\}\Big/F(2w,2K_x,2K_y) \tag{16}$$

$$\gamma_1=(w-K_xU_1+iK^2D)^{-1}$$
$$\gamma_2=(w-K_xU_1+2iK^2D)^{-1}$$
$$\gamma_3=(w-K_xU_1-iK^2D)^{-1}$$

It is interesting to observe that equation (14) has a singularity in Λ for $F(2w, 2K_x, 2K_y) = 0$. This corresponds to the case of second harmonic resonance, where the fundamental resonance cannot exist without the presence of its first harmonic. We should remark that the analysis given in this paper is not valid in the neighborhood of such resonance.

3. DERIVATION OF THE GINZBURG–LANDAU EQUATION

We shall study equation (14) by the nonvanishing of the first derivatives of F. In this case, equation (13) can be rewritten as

$$\frac{\partial A}{\partial t_1} + \frac{\partial w}{\partial K_x}\frac{\partial A}{\partial x_1} + \frac{\partial w}{\partial K_y}\frac{\partial A}{\partial y_1} = 0 \tag{17}$$

where

$$\frac{\partial w}{\partial K_x} = -\frac{\partial F}{\partial K_x}\bigg/\frac{\partial F}{\partial w} \quad \text{and} \quad \frac{\partial w}{\partial K_y} = -\frac{\partial F}{\partial K_y}\bigg/\frac{\partial F}{\partial w}$$

Substituting (17) into (14), we obtain the following partial differential equation using the original variables x, y, and t:

$$i\frac{\partial A}{\partial T} + \frac{1}{2}\left(\frac{\partial^2 w}{\partial K_x^2}\frac{\partial^2 A}{\partial X^2} + 2\frac{\partial^2 w}{\partial K_x \partial K_y}\frac{\partial^2 A}{\partial X \partial Y} + \frac{\partial^2 w}{\partial K_y^2}\frac{\partial^2 A}{\partial Y^2}\right) = -\left(J\bigg/\frac{\partial F}{\partial w}\right)A^2\bar{A} \tag{18}$$

where

$$X = \varepsilon\left(x - \frac{\partial w}{\partial K_x}t\right), \qquad Y = \varepsilon\left(y - \frac{\partial w}{\partial K_y}t\right), \qquad T = \varepsilon^2 t$$

By introducing the transformations $\xi = lX + mY$ and $\tau = T$, where l and m are arbitrary constants, we reduce equation (18) to

$$\frac{\partial A}{\partial \tau} = (P_r + iP_i)\frac{\partial^2 A}{\partial \xi^2} + (Q_r + iQ_i)A^2\bar{A} \tag{19}$$

where

$$P_r + iP_i = \frac{i}{2}\left(l^2\frac{\partial^2 w}{\partial K_x^2} + 2lm\frac{\partial^2 w}{\partial K_x \partial K_y} + m^2\frac{\partial^2 w}{\partial K_y^2}\right)$$

and

$$Q_r + iQ_i = iJ\bigg/\frac{\partial F}{\partial w}$$

Equation (19) is the well-known Ginzburg–Landau equation. It is also known that the solutions of equation (19) are stable if and only if (Lange and Newell, 1974)

$$P_rQ_r + P_iQ_i < 0 \tag{20}$$

and

$$Q_r < 0 \tag{21}$$

Landman (1987) studied a particular class of solutions of this equation of the form

$$A(\xi, \tau) = e^{-i\Omega\tau} f(\xi - c\tau)$$

which he called quasisteady solutions, and found that their spatial variation may be periodic, quasiperiodic, or apparently chaotic. Also, Rotenberry and Saffman (1990) used equation (19) to study the weakly nonlinear two-dimensional evolution of a disturbance in a channel with compliant walls for Reynolds number near its critical value.

We now return once more to equation (19) to study the following two cases of physical interest.

3.1. The Rayleigh–Taylor Instability

The classical Rayleigh–Taylor problem (no mean flow) treats the stability of a dense fluid overlaying a less dense fluid.

The following simplifications are imposed on equations (12) and (19), corresponding to the absence of mean flow ($U_1 = U_2 = 0$) and the presence of a tangential field.

The linear dispersion relation, which is obtained from equation (12), is

$$a_0(-iw)^3 + a_1(-iw)^2 + a_2(-iw) + a_3 = 0 \tag{22}$$

where

$$a_0 = \rho_1 + \rho_2 + \Gamma_0 K$$

$$a_1 = DK^2 a_0$$

$$a_2 = gK(\rho_1 - \rho_2) + g\Gamma_0 K^2 + B_0^2 K_x^2 + \sigma_0 K^3$$

$$a_3 = DK^2[gK(\rho_1 - \rho_2) + B_0^2 K_x^2 + \sigma_0 K^3]$$

with

$$B_0^2 = \frac{(\mu_2 - \mu_1)^2}{4\pi(\mu_2 + \mu_1)} H_0^2$$

We know from the Routh–Hurwitz criterion that the necessary and sufficient conditions for stability are

$$a_1 > 0, \qquad a_2 > 0, \qquad a_3 > 0, \qquad a_1 a_2 / a_0 a_3 > 1 \tag{23}$$

since a_0 is always positive. Therefore, the system is linearly stable if and only if $B_0 > B_c$, where B_c is given by the relation (for $K = K_x$)

$$B_c^2 = [g(\rho_2 - \rho_1) - \sigma_0 K^2]K \tag{24}$$

We observe that the magnetic field has a stabilizing influence on the wave motion. These theoretical results were first obtained and confirmed experimentally by Zelazo and Melcher (1969) (see also Rosensweig, 1985).

In the nonlinear theory, equation (19) is reduced to the nonlinear diffusion equation

$$\frac{\partial A}{\partial \tau} = P_r \frac{\partial^2 A}{\partial \xi^2} + Q_r A^2 \bar{A} \tag{25}$$

where

$$P_r = -\frac{D}{g^3\Gamma_0^3}\{g^2\Gamma_0^2(3\sigma_0 K + B_c^2) + [g\Gamma_0 - K^2D^2(\rho_1 + \rho_2 + \Gamma_0 K)](2\sigma_0 K + B_c^2)^2\} \tag{26}$$

$$Q_r = \frac{DK^4}{g\Gamma_0}\left\{K\left(g\Gamma_0 - 2B_c^2\frac{1-\mu}{1+\mu}\right)\left(g\Gamma_0 - B_c^2\frac{1-\mu}{1+\mu}\right) \times [g(\rho_2 - \rho_1) + 2\sigma_0 K^2]^{-1} + 1.5\sigma_0 K + 2B_c^2\right\} \tag{27}$$

The coefficients P_r and Q_r are evaluated when $U_1 = U_2 = 0$. In this case the coefficients P_i and Q_i are equal to zero.

The stability conditions of equation (25) are

$$P_r > 0 \quad \text{and} \quad Q_r < 0 \tag{28}$$

The stability can therefore be discussed by dividing the $B_c^2 - K$ plane into stable and unstable regions. The transition curves are given by the vanishing of P_r and Q_r. The curves are

$$g^2\Gamma_0^2(3\sigma_0 K + B_c^2) + (2\sigma_0 K + B_c^2)^2[g\Gamma_0 - K^2D^2(\rho_1 + \rho_2 + \Gamma_0 K)] = 0 \tag{29}$$

and

$$K[g(\rho_2 - \rho_1) + 2\sigma_0 K^2]^{-1}[g\Gamma_0 - 2B_c^2(1-\mu)/(1+\mu)] \times [g\Gamma_0 - B_c^2(1-\mu)/(1+\mu)] + 1.5\sigma_0 K + 2B_c^2 = 0 \tag{30}$$

We observe that Q_r changes sign at

$$K^2 = g(\rho_1 - \rho_2)/2\sigma_0 \tag{31}$$

which is the second harmonic resonance.

In what follows, we shall discuss the stability by drawing the curves in the $B_c^2 - K$ plane for various values of σ_0. In the following graphs the dashed curve represents the linear stability curve, which is assumed to divide the plane into a stable region S above the curve and an unstable region U below the curve in the linear sense. The shaded regions are newly formed regions due to the nonlinear effects. U_1 is an unstable region, while S_1 is a stable region.

Figure 1 represents a system when the influence of surface tension is neglected ($\sigma_0 = 0$). The graph is divided into stable regions (S, S_1) and unstable regions (U, U_1) according to equations (24) and (29)–(31). We observe that the newly formed region U_1 occurs for large values of the field and for a band of wavenumbers less than unity, approximately. We observe also that the region S_1 is characterized by relatively smaller values of the field within a band of values of the field and extends for arbitrary wavenumbers. A large value of the field requires a smaller value of the wavenumber to achieve stability in the region S_1. Finally, the curve given by equation (31) does not appear because $\rho_1 < \rho_2$ in our system.

Figure 2 represents the same system as considered in Fig. 1, but for $\sigma_0 = 20$ dyne/cm. We observe a similar behavior to that in Fig. 1, but the region S_1 extends downward because the linear curve is far from the curve $Q_r = 0$.

Figure 3 represents the same system as considered in Fig. 1, but having the fluids interchanged, $\Gamma_0 = 0.001$ and $\sigma_0 = 1.44$ dyne/cm. This means that $\rho_2 < \rho_1$ and consequently the linear curve does not appear in the graph [see equation (24)]. Therefore the system is linearly stable. The effect of nonlinearity produces a curve showing that an unstable region exists below

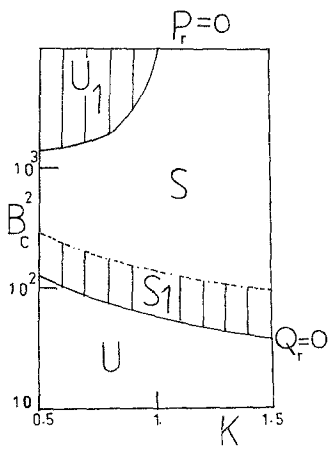

Fig. 1. The system whose particulars are $\rho_2 = 1.064$ g/cm^3, $\rho_1 = 0.9142$ g/cm^3, $g = 981$ cm/sec^2, $D = 1.8 \times 10^{-5}$ cm^2/sec, $\mu = 1/6$, $\Gamma_0 = 20$ g/cm^2, and $\sigma_0 = 0$ dyne/cm. The dashed line represents the linear curve. The figure is computed from the relations (24) and (29)–(31) to indicate the transition from stability to instability.

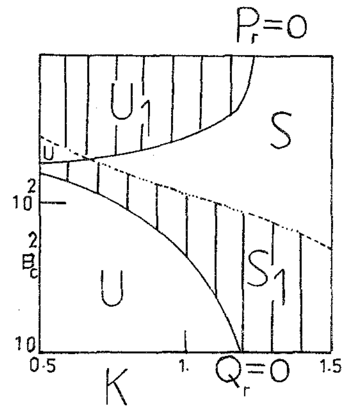

Fig. 2. The same system as in Fig. 1, but with $\sigma_0 = 20$ dyne/cm.

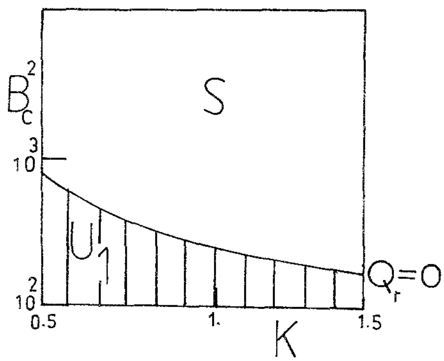

Fig. 3. The same system as in Fig. 1, but interchanged, $\Gamma_0 = 0.001$ g/cm^2 and $\sigma_0 = 1.44$ dyne/cm.

the curve. Finally, the curve representing second harmonic resonance appears at $K = 7.14$ approximately.

3.2. The Kelvin–Helmholtz Instability

Classical Kelvin–Helmholtz instability relates to the behavior of a plane interface between moving fluid layers. A basic situation in ferrohydrodynamics is the inviscid wave behavior at the interface between layers of magnetized fluids having permeabilities μ_1 and μ_2.

Let us now consider the case where the primary flow state is given by two uniform streams moving with uniform velocities U_1 and U_2 in the x direction.

Proceeding as in the previous case, we find that the dispersion relation, with complex coefficients, for the Kelvin–Helmholtz problem is

$$A_0 w^3 + A_1 w^2 + A_2 w + A_3 = 0 \tag{32}$$

where

$$A_0 = \rho_1 + \rho_2 + \Gamma_0 K$$

$$A_1 = -K_x\{A_0 U_1 + 2[U_1(\rho_1 + \Gamma_0 K) + \rho_2 U_2]\} + iK^2 D A_0$$

$$\begin{aligned} A_2 = {} & K_x^2\{U_1^2(\rho_1 + \Gamma_0 K) + \rho_2 U_2^2 + 2U_1[U_1(\rho_1 + \Gamma_0 K) + \rho_2 U_2]\} \\ & - gK(\rho_1 - \rho_2) - g\Gamma_0 K^2 - B_0^2 K_x^2 - \sigma_0 K^3 - 2iK^3 D \\ & \times [U_1(\rho_1 + \Gamma_0 K) + \rho_2 U_2] \end{aligned}$$

$$\begin{aligned} A_3 = {} & K_x U_1\{gK(\rho_1 - \rho_2) + g\Gamma_0 K^2 + B_0^2 K_x^2 + \sigma_0 K^3 \\ & - K_x^2[U_1^2(\rho_1 + \Gamma_0 K) + \rho_2 U_2^2]\} + iK^2 D\{K_x^2[U_1^2(\rho_1 + \Gamma_0 K) + \rho_2 U_2^2] \\ & - [gK(\rho_1 - \rho_2) + B_0^2 K_x^2 + \sigma_0 K^3]\} \end{aligned}$$

It can be easily shown that all the imaginary parts of all the roots of equation (32) are negative if and only if (Zahreddine and El Shehawey, 1988)

$$\begin{gathered} A_0 > 0, \qquad A_{1i} > 0 \\ A_{1i}(A_0 A_{3i} - A_{1i} A_{2r}) + A_{2i}(A_{1r} A_{1i} - A_0 A_{2i}) > 0 \\ [A_{2i}(A_{1r} A_{1i} - A_0 A_{2i}) + A_{1i}(A_0 A_{3i} - A_{1i} A_{2r})][A_{3i}(A_{2r} A_{1i} - A_0 A_{3i}) \\ - A_{1i} A_{2i} A_{3r}] - [A_{3r} A_{1i}^2 - A_{3i}(A_{1r} A_{1i} - A_0 A_{2i})]^2 > 0 \end{gathered} \tag{33}$$

where

$$A_m = A_{mr} + iA_{mi}, \qquad m = 1, 2, 3$$

We notice that the first three conditions are trivially satisfied. But the fourth condition gives, for $B_0 > \tilde{B}_c$,

$$\tilde{B}_c^2 = \frac{g(\rho_2 - \rho_1)}{K} + \rho_2(U_1 - U_2)^2 - \sigma_0 K \tag{34}$$

It is obvious that the system is linearly stable for $B_0 > \tilde{B}_c$. The relation (34) shows similar behavior as that of the Rayleigh–Taylor instability. The Kelvin–Helmholtz instability, however, requires larger values of the magnetic field than those in the Rayleigh–Taylor instability in order to achieve stability. The reason is due to the destabilizing nature of the Kelvin–Helmholtz flow.

A nonlinear analysis of the Kelvin–Helmholtz instability will be studied by employing equations (19)–(21), where

$$P_r + iP_i = \frac{i}{2}\left[2\rho_2(U_1 - U_2) + \frac{ig\Gamma_0}{KD}\right]^{-3}\left\{\left[2\rho_2(U_1 - U_2) + \frac{ig\Gamma_0}{KD}\right]^2 \right.$$
$$\times\left[2\sigma_0 - \frac{2U_1^2}{K}\left(\rho_1 + \rho_2 + \Gamma_0 K - \frac{g\Gamma_0}{K^2D^2}\right) - \frac{2ig\Gamma_0 U_1}{K^2D}\right]$$
$$+2\left[2\rho_2(U_1 - U_2) + \frac{ig\Gamma_0}{KD}\right]\left[\frac{2U_1}{K}\left(\rho_1 + \rho_2 + \Gamma_0 K - \frac{g\Gamma_0}{K^2D^2}\right) + \frac{ig\Gamma_0}{K^2D}\right]$$
$$\times\left[2\sigma_0 K + \rho_2(U_1^2 - U_2^2) + \tilde{B}_c^2 + \frac{ig\Gamma_0 U_1}{KD}\right]$$
$$-\frac{2}{K}\left(\rho_1 + \rho_2 + \Gamma_0 K - \frac{g\Gamma_0}{K^2D^2}\right)$$
$$\left.\times\left[2\sigma_0 K + \rho_2(U_1^2 - U_2^2) + \tilde{B}_c^2 + \frac{ig\Gamma_0 U_1}{KD}\right]^2\right\} \tag{35}$$

and

$$Q_r + iQ_i = -iK^3\left[2\rho_2(U_1 - U_2) + \frac{ig\Gamma_0}{KD}\right]^{-1}$$
$$\times\left\{K[g(\rho_1 - \rho_2) - 2\sigma_0 K^2]^{-1}\left[g\Gamma_0 + 2\rho_2(U_1 - U_2)^2 + 2\tilde{B}_c^2\frac{\mu - 1}{\mu + 1}\right]\right.$$
$$\times\left[g\Gamma_0 + \rho_2(U_1 - U_2)^2 + \tilde{B}_c^2\frac{\mu - 1}{\mu + 1}\right]$$
$$\left. + [2\rho_2(U_1 - U_2)^2 - 1.5\sigma_0 K - 2\tilde{B}_c^2]\right\} \tag{36}$$

As mentioned above, the transition curves are given by, accordingly, the conditions (20) and (21)

$$P \cdot Q = 0, \qquad Q_r = 0 \tag{37}$$

and the curve indicates subharmonic resonance, where $P \cdot Q = P_rQ_r + P_iQ_i$.

The stability analysis may be understood by studying the stability graphs represented by equations (34)–(37).

Figure 4 represents a system at initially zero surface tension ($\sigma_0 = 0$). Two unstable regions U_1 and U_2 appear, which results in reducing the nonlinearly stable region S. Thus the effect of nonlinearity is destabilizing except at a very small region S_1 characterized by small wavenumbers less than unity and a limited range of the field.

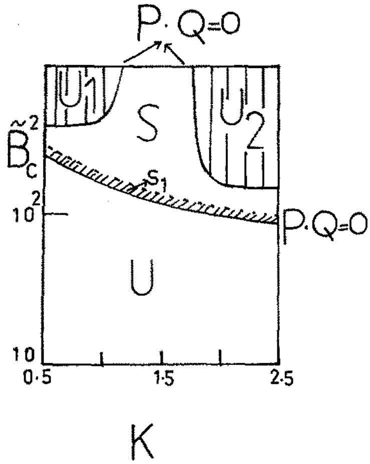

Fig. 4. The same system as in Fig. 1, with $U_1 = 1$ cm/sec and $U_2 = 7$ cm/sec. The figure is computed from the relations (34)–(37) to indicate the transition curves.

Figure 5 represents the same system as in Fig. 4, but $\sigma_0 = 20$ dyne/cm. We observe that the region S_1 is enlarged to cover a wide range of wavenumbers between $K = 1.6$ to $K = 2.2$ covering a wider range of the applied field. Out this range of K the surface tension is destabilizing in the nonlinear sense. Thus the surface tension plays a dual role.

Figures 6–8 represent a system for three different values of $\tilde{B}_c^2$ and the influence of gravity is neglected ($g = 0$).

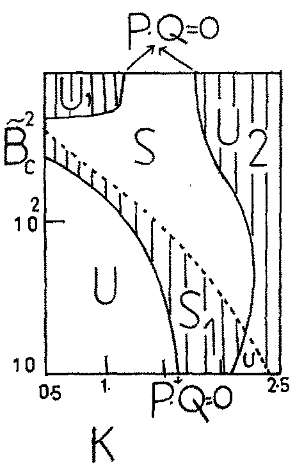

Fig. 5. The same system as in Fig. 4, but $\sigma_0 = 20$ dyne/cm.

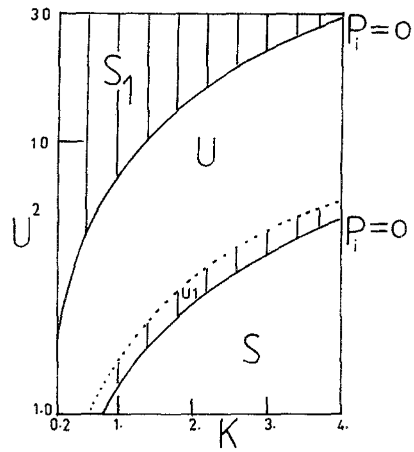

Fig. 6. Stability diagram in the U_c^2-K plane, where $U_1 = U_c$, $U_2 = 0$, $g = 0$, $\Gamma_0 = 0.1$, $\sigma_0 = 1.44$, $\rho_2 = 0.914$, $\rho_1 = 1.064$, and $\tilde{B}_c^2 = 0$.

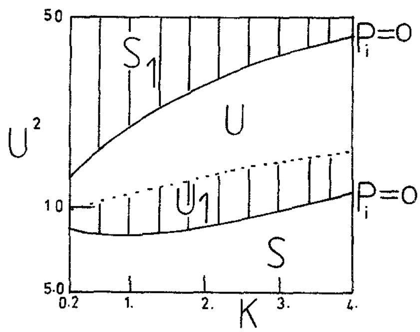

Fig. 7. Stability diagram for the same system as considered in Fig. 6, but with $\tilde{B}_c^2 = 9$.

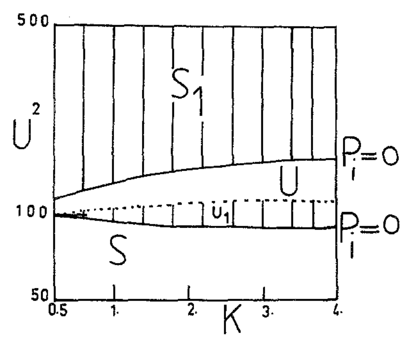

Fig. 8. Stability diagram for the same system as considered in Fig. 6, but with $\tilde{B}_c^2 = 99$.

In this case $P_r = Q_r = 0$, and equation (19) is reduced to the nonlinear Schrödinger equation,

$$i\frac{\partial A}{\partial \tau} + P_i\frac{\partial^2 A}{\partial \xi^2} + Q_i A^2\bar{A} = 0 \tag{38}$$

where

$$\begin{aligned} P_i = \{&4\rho_2^2(U_1-U_2)^2[\sigma_0 - U_1^2(\rho_1+\rho_2+\Gamma_0 K)/K] \\ &+ (4/K)U_1\rho_2(U_1-U_2)(\rho_1+\rho_2+\Gamma_0 K)[2\sigma_0 K + \rho_2(U_1^2-U_2^2) + \tilde{B}_c^2] \\ &- (\rho_1+\rho_2+\Gamma_0 K)[2\sigma_0 K + \rho_2(U_1^2-U_2^2)+\tilde{B}_c^2]^2/K\}/[8\rho_2^3(U_1-U_2)^3] \end{aligned}$$

and

$$\begin{aligned} Q_i = [K^3/\rho_2(U_1-U_2)]\{&[\rho_2(U_1-U_2)^2 + \tilde{B}_c^2(\mu-1)/(\mu+1)]^2/(2\sigma_0 K) \\ &+ \frac{3}{4}\sigma_0 K + \tilde{B}_c^2 - \rho_2(U_1-U_2)^2\} \end{aligned}$$

It is well known that the solutions of equation (38) are unstable if $P_iQ_i > 0$. Therefore, we observe that the unstable regions (U, U_1) decrease with the increase of the magnetic field. We also observe that the region S increases with the increase of the field, while the new stable region S_1 decreases with the increase of the field.

4. STABILITY OF SOLITARY WAVES

The analysis of this section will be based on equation (19) when the surface tension is constant. In this case, the coefficients P_r and Q_r are equal to zero. Therefore, the solutions of this equation are stable if and only if $P_iQ_i < 0$. The parameters P_i and Q_i can take positive or negative values.

The simplest localized solution to equation (19) for $P_iQ_i > 0$ is called an envelope soliton and is expressed as

$$A = \lambda_0^{1/2}\,\mathrm{sech}[\xi(-\lambda_0 Q_i/2P_i)^{1/2}]\exp(-i\Omega_0\tau) \tag{39}$$

where λ_0 is constant and $\Omega_0 = 0.5\lambda_0 Q_i$.

The stability of the envelope soliton was first studied by Zakharov (1968), who established that the solution is unstable. This result was generalized by Zakharov and Rubenchik (1974) by studying perturbations about marginally stable states.

For $P_iQ_i < 0$, the solution to equation (19) can be written in the form

$$A = \lambda^{1/2}\left[1 - \mu^2\,\mathrm{sech}^2\left(\left|\frac{Q_i}{2P_i}\right|^{1/2}\lambda\mu\xi\right)\right]^{1/2}\exp[i\nu(\xi,\tau)] \tag{40}$$

with

$$\mu^2 = (\lambda^2 - \lambda_{\min}^2)/\lambda^2 \leq 1$$

and

$$v = \sin^{-1}\left\{\frac{\mu \tanh(|Q_i/2P_i|^{1/2}\lambda\mu\xi)}{[1 - \mu^2 \operatorname{sech}^2(|Q_i/2P_i|^{1/2}\lambda\mu\xi)]^{1/2}}\right\} - \Omega t$$

where $\Omega = 0.5\lambda^2(\mu^2 - 3)$ is the nonlinear frequency shift. The above solution represents an envelope hole. Compared with the envelope soliton, the envelope hole has an additional independent parameter μ, which designates the depth of modulation. When $\mu = 1$, $\lambda_{\min}$ reaches zero and the solution can be expressed as an envelope shock.

Studies of solitary waves are well known, and in particular the solution corresponding to that given by equation (14). It is known that an initial wave packet of an arbitrary envelope will disintegrate into a number of envelope solitons and oscillatory tail. Moreover, the solutions of equation (19) for periodic boundary conditions satisfy the Fermi–Pasta–Ulam recurrence phenomenon (Newell, 1983). This means that a solitary wave reaches a maximum modulation and eventually returns to an unmodulated state.

5. CONCLUSIONS

We applied the multiple-scales method to the Rayleigh–Taylor and Kelvin–Helmholtz stability problems of two incompressible, inviscid, magnetic fluids, taking into account the surface tension effect, to derive a linear and a nonlinear evolution of the modulational instability.

For the case of the linear Rayleigh–Taylor stability problem, it is found that both the surface tension and the magnetic field are strictly stabilizing, while the adsorption has no effect. Also, the ratio of the magnetic permeability has no effect in the sense that it does not matter which of the fluids has a larger permeability constant.

For the case of the linear Kelvin–Helmholtz stability problem, however, the classical stability criterion is found to be substantially modified due to the effect of the tangential field. When $U_1 = U_2 = U$, the real part of w is the same as that of the Rayleigh–Taylor case, except for an additive term $K_x U$ to take care of the streaming of the fluid. Thus when $|U_1 - U_2|$ is small, the behavior of the flow system differs slightly from the Rayleigh–Taylor case.

In the nonlinear theory, we obtain a Ginzburg–Landau equation for the Kelvin–Helmholtz model and a nonlinear diffusion equation for the Rayleigh–Taylor model to describe the behavior of the disturbed system and discuss their stability conditions. From these equations and their stability conditions, we notice that the nonlinear interaction coefficient depends on the sign of $\mu_1 - \mu_2$. Therefore the stability criterion will strongly depend on whether the lower fluid or the upper fluid has a greater magnetic permeability. The results are in contrast with the linear theory, where the sign of $\mu_1 - \mu_2$ has no implications for the stability criterion.

A similar phenomenon was observed in the surface instability of dielectric fluids in vertical electric fields. Owing to the nonlinear effects, Mohamed and El Shehawey (1983) found that the instability depends on the dielectric constant difference.

The tangential field plays a dual role in the stability criterion of the system. The results are illustrated in the stability charts of the system. It is found that under certain conditions, the destabilizing effect of streaming can be suppressed by a suitable choice of the field and vice versa.

REFERENCES

Elhefnawy, A. R. F. (1990). *Journal of Applied Mathematics and Physics (ZAMP)*, **41**, 669–683.

Funada, T. (1987). *Journal of the Physical Society of Japan*, **56**, 2031–2038.

Landman, M. J. (1987). *Studies in Applied Mathematics*, **76**, 187–237.

Lange, C. C., and Newell, A. C. (1974). *SIAM Journal of Applied Mathematics*, **27**, 441–456.

Levich, V. G. (1962). *Physico-Chemical Hydrodynamics*, Prentice-Hall, Englewood Cliffs, New Jersey.

Malik, S. K., and Singh, M. (1986). *Physics of Fluids*, **29**, 2853–2859.

Mohamed, A. A., and El Shehawey, E. F. (1983). *Journal of Fluid Mechanics*, **129**, 473–494.

Nayfeh, A. H. (1976). *Journal of Applied Mechanics*, **98**, 584–588.

Newell, A. C. (1983). *Journal of Applied Mechanics*, **105**, 1127–1138.

Oron, A., and Rosenau, P. (1992). *Journal of Physics II France*, **2**, 131–146.

Pearson, J. R. A. (1958). *Journal of Fluid Mechanics*, **4**, 489–500.

Rosensweig, R. E. (1985). *Ferrohydrodynamics*, Cambridge University Press, Cambridge.

Rotenberry, J. M., and Saffman, P. G. (1990). *SIAM Journal of Applied Mathematics*, **50**, 361–394.

Scriven, L. E., and Sternling, C. V. (1960). *Nature*, **187**, 186–188.

Sivashinsky, G. I. (1982). *Physica D*, **4**, 227–235.

Sørensen, T. S. (1978). Instabilities induced by mass transfer, low surface tension and gravity at isothermal and deformable fluid interfaces in dynamics and instability of fluid interfaces, in *Lecture Notes in Physics*, Vol. 105, pp. 1–74.

Zahn, M., and Rosensweig, R. E. (1991). *Advances in Porous Media*, M. Y. Corapciologlu, ed., Vol. 1, pp. 125–178.

Zahreddine, Z., and El Shehawey, E. F. (1988). *Indian Journal of Pure and Applied Mathematics*, **19**, 963–972.

Zakharov, V. E. (1968). *Soviet Physics JETP*, **26(4)**, 994–998.

Zakharov, V. E., and Rubenchik, A. M. (1974). *Soviet Physics JETP*, **38**, 494–500.

Zelazo, R. E., and Melcher, J. R. (1969). *Journal of Fluid Mechanics*, **39**, 1–24.